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THICKNESS–TWIST VIBRATIONS OF AN INFINITE, MONOCLINIC, CRYSTAL PLATE

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Abstract—Thickness–twist modes of vibration of an infinite plate are standing waves with displacement and wave-normal at right angles to each other and parallel to the faces of the plate. In a monoclinic crystal plate, such waves are possible if the displacement is in the direction of the digonal axis. The pertinent solution of the three-dimensional equations of elasticity for a plate with free faces is given, as well as another solution with the effect of the inertia of electrode films taken into account. The results are applicable to the rotated-*Y*-cut quartz plates used as oscillator and filter crystals in electric circuits.

INTRODUCTION

INTEREST in the frequencies of vibration of monoclinic, crystal plates, stems from the fact that the widely used rotated-*Y*-cut quartz plates [1] exhibit monoclinic symmetry in the stress–strain relation when it is referred to rectangular axes in and normal to the plane of the plate. Thus, with strain–displacement relations

$$\begin{aligned} S_1 &= \frac{\partial u}{\partial x}, & S_4 &= \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z},\\ S_2 &= \frac{\partial v}{\partial y}, & S_5 &= \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\\ S_3 &= \frac{\partial w}{\partial z}, & S_6 &= \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}, \end{aligned} \tag{1}$$

the stress–strain relation for a rotated-*Y*-cut plate of quartz, with x the digonal axis, is

$$\begin{aligned} T_1 &= c_{11}S_1+c_{12}S_2+c_{13}S_3+c_{14}S_4,\\ T_2 &= c_{21}S_1+c_{22}S_2+c_{23}S_3+c_{24}S_4,\\ T_3 &= c_{31}S_1+c_{32}S_2+c_{33}S_3+c_{34}S_4,\\ T_4 &= c_{41}S_1+c_{42}S_2+c_{43}S_3+c_{44}S_4,\\ T_5 &= c_{55}S_5+c_{56}S_6,\\ T_6 &= c_{65}S_5+c_{66}S_6. \end{aligned} \tag{2}$$

The stresses, in turn, must satisfy the equations of motion

$$\frac{\partial T_1}{\partial x}+\frac{\partial T_6}{\partial y}+\frac{\partial T_5}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial T_6}{\partial x}+\frac{\partial T_2}{\partial y}+\frac{\partial T_4}{\partial z} = \rho\frac{\partial^2 v}{\partial t^2},$$
$$\frac{\partial T_5}{\partial x}+\frac{\partial T_4}{\partial y}+\frac{\partial T_3}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}. \tag{3}$$

The problem at hand is the study of the steady state solutions of the fifteen equations (1)–(3) subject to the boundary conditions

$$T_2 = T_4 = T_6 = 0 \qquad \text{on} \qquad y = \pm h. \tag{4}$$

Detailed analyses have been made [2,3] of the highly complicated solution for the case of the wave-normal parallel to the digonal axis. There is, however, a much simpler solution for waves with normal in the direction of z and displacement in the direction of x. Moreover, the simple solution is one of technological interest as it involves only the strains (S_5 and S_6) that are linked directly, through the piezoelectric relations [1], to the electric field (E_2) applied via electrodes on the surfaces $y = \pm h$.

SOLUTION

Consider the possibility of a solution of equations (1)–(3) of the form

$$u = U(y, z)e^{i\omega t}, \qquad v = w = 0. \tag{5}$$

Then, with the exponential factor omitted, equations (1) reduce to

$$S_1 = S_2 = S_3 = S_4 = 0,$$
$$S_5 = \frac{\partial U}{\partial z}, \qquad S_6 = \frac{\partial U}{\partial y}; \tag{6}$$

equations (2), with (6), reduce to

$$T_1 = T_2 = T_3 = T_4 = 0,$$
$$T_5 = c_{55}\frac{\partial U}{\partial z}+c_{56}\frac{\partial U}{\partial y},$$
$$T_6 = c_{56}\frac{\partial U}{\partial z}+c_{66}\frac{\partial U}{\partial y}; \tag{7}$$

and equations (3), with (7), reduce to

$$c_{66}\frac{\partial^2 U}{\partial y^2}+2c_{56}\frac{\partial^2 U}{\partial y\,\partial z}+c_{55}\frac{\partial^2 U}{\partial z^2} = -\rho\omega^2 U. \tag{8}$$

In particular,

$$U = A\sin\eta y\cos\zeta z+B\cos\eta y\sin\zeta z \tag{9}$$

is a solution of (8) if

$$B = \pm A, \qquad \rho\omega^2 = c_{66}\eta^2+c_{55}\zeta^2\pm 2c_{56}\eta\zeta.$$

Thus, two solutions of the form (9) are possible. With a view to satisfying the boundary conditions (4), take a sum of the two solutions:

$$U = C_1 \sin(\eta_1 y + \zeta z) + C_2 \sin(\eta_2 y - \zeta z). \tag{10}$$

Then (8) requires

$$\rho\omega^2 = c_{66}\eta_1^2 + c_{55}\zeta^2 + 2c_{56}\eta_1\zeta = c_{66}\eta_2^2 + c_{55}\zeta^2 - 2c_{56}\eta_2\zeta, \tag{11}$$

from which

$$c_{66}(\eta_1^2 - \eta_2^2) + 2c_{56}\zeta(\eta_1 + \eta_2) = 0.$$

Hence, either $\eta_1 + \eta_2 = 0$ or

$$c_{66}(\eta_2 - \eta_1) = 2c_{56}\zeta. \tag{12}$$

Now, with (5), the boundary conditions (4) reduce to

$$c_{56}\frac{\partial U}{\partial z} + c_{66}\frac{\partial U}{\partial y} = 0 \qquad \text{on} \qquad y = \pm h. \tag{13}$$

Then, with (10) and (12), (13) reduce to

$$\frac{C_2}{C_1} = -\frac{\cos \eta_1 h}{\cos \eta_2 h} = \frac{\sin \eta_1 h}{\sin \eta_2 h}, \tag{14}$$

whence

$$\sin(\eta_1 + \eta_2)h = 0, \tag{15}$$

or

$$(\eta_1 + \eta_2)h = m\pi; \qquad m = 0, 1, 2, 3, \ldots. \tag{16}$$

From (16) and (12),

$$\eta_1 = \frac{m\pi}{2h} - \frac{c_{56}}{c_{66}}\zeta, \qquad \eta_2 = \frac{m\pi}{2h} + \frac{c_{56}}{c_{66}}\zeta$$

and then (11) becomes

$$\omega^2 = \frac{c_{66}}{\rho}\left(\frac{\pi}{2h}\right)^2 \left[m^2 + \frac{\gamma_{55}}{c_{66}}\left(\frac{2\zeta h}{\pi}\right)^2\right], \tag{17}$$

where $\gamma_{55} = c_{55} - c_{56}^2/c_{66}$.

The dispersion relation (17) is plotted in Fig. 1 with the ordinate the dimensionless ratio of the frequency, ω, to the frequency, ω_s, of the lowest $x-y$ thickness-shear mode:

$$\omega_s = \frac{\pi}{2h}\sqrt{\frac{c_{66}}{\rho}}.$$

The branches for real, positive ζ are quarter-rectangular hyperbolas and for imaginary, positive ζ are quarter-circles for the chosen dimensionless ordinate and abscissa.

Effect of inertia of electrodes

Suppose that each face $y = \pm h$ of the plate is coated with a thin electrode film of

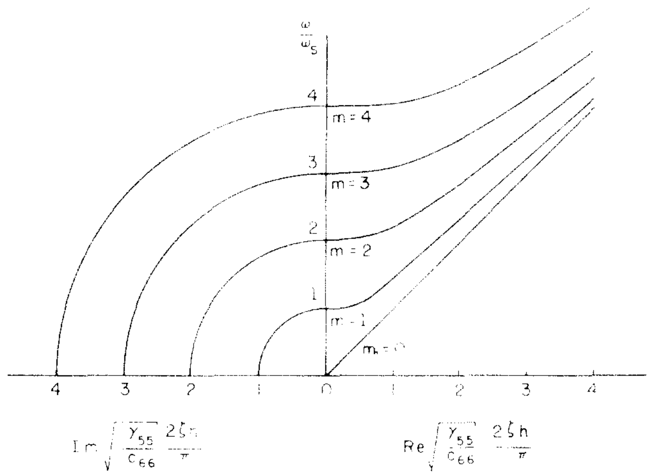

FIG. 1. Dispersion curves for thickness–twist modes of an infinite, monoclinic, crystal plate.

density ρ' and thickness $2h' \ll h$. The inertia of the electrodes is taken into account by replacing the boundary conditions (13) with

$$c_{56}\frac{\partial U}{\partial z}+c_{66}\frac{\partial U}{\partial y} = \pm 2\rho' h'\omega^2 U \qquad \text{on} \qquad y = \pm h, \tag{18}$$

which are satisfied by (10) if

$$\frac{C_2}{C_1} = -\frac{k\cos\eta_1 h-\sin\eta_1 h}{k\cos\eta_2 h-\sin\eta_2 h} = \frac{k\sin\eta_1 h+\cos\eta_1 h}{k\sin\eta_2 h+\cos\eta_2 h}, \tag{19}$$

where

$$k = \frac{c_{66}\eta_1+c_{56}\zeta}{2\rho' h'\omega^2} = \frac{c_{66}\eta_2-c_{56}\zeta}{2\rho' h'\omega^2}.$$

From (19),

$$\tan(\eta_1+\eta_2)h = 2k/(1-k^2), \tag{20}$$

which replaces (15) and, along with

$$(\eta_2-\eta_1)c_{66} = 2c_{56}\zeta, \tag{21}$$

$$\rho\omega^2 = c_{66}\eta_1^2+c_{55}\zeta^2+2c_{56}\eta_1\zeta, \tag{22}$$

constitutes the dispersion relation for thickness–twist waves in the plate with mass loading of electrodes accounted for.

Equations (20)–(22) give frequencies only slightly lower than (17) as the ratio

$$R = 2\rho' h/\rho h \tag{23}$$

of the mass per unit area of electrodes to the mass per unit area of plate is usually only of the order of 10^{-2}. For example, at cut-off ($\zeta = 0$), (20)–(22) reduce to

$$\omega/\omega_s = 2\eta h/\pi, \tag{24}$$

$$\tan 2\eta h = 2R\eta h/(R^2\eta^2h^2-1). \tag{25}$$

The roots of technological interest give frequencies near those of (17) for $\zeta = 0$ and m a small, odd integer. With

$$2\eta h = m\pi - \varepsilon, \qquad 0 < \varepsilon \ll 1,$$

equation (25) gives, approximately,

$$2\eta h = m\pi(1-R).$$

Hence, the cut-off frequencies for a plate with electrodes are approximately

$$\omega/\omega_s = m(1-R),$$

for small R and m, instead of $\omega/\omega_s = m$ which was found for the plate without electrodes.

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Zusammenfassung—Bei den torsionalen Dickenschwingungen einer unbegrenzten Platte entstehen parallel zu den Plattenflächen verlaufende Stehwellen, deren Auslenkung und Wellenfrontnormale rechtwinklig zueinander stehen. In einem monoklinischem Kristall sind solche Wellen dann gegeben, wenn die Auslenkung in die Richtung der Digonalachse fällt. Eine geeignete Lösung der dreidimensionalen Elastizitätsgleichungen und ebenfalls eine andere Lösung, welche den Trägheitseinfluss des Elektrodenbelags in Betracht zieht, wird für eine Platte mit freien Flächen angegeben. Die Ergebnisse sind auf die in elektrischen Schaltungen als Oszillatoren und Filterkristalle verwendeten Quarzplättchen mit verdrehtem Y-Schnitt anwendbar.

Абстракт—Формы колебаний, зависящих от толщины и кручения бесконечной плиты, являются стоячими волнами, со смещением и главной волной перпендикулярными между собой и параллельными к поверхностям плиты. В моноклинической кристаллической плите такие волны возможны, если смещение имеет место в направлении диагональной оси. Дается подходящее решение пространственных уравнений упругости для плиты со свободными поверхностями; также дается другое решение, в котором принят во внимание эффект инерции электродных фильм. Результаты могут быть применяемы для вращаемой, срезанной под острым углом кварцевой плиты, употребляемой как осциллятор и как фильтр-кристалл в электрических схемах.